TOPOLOGY HOMEWORK 3, DUE SEPTEMBER 21

Instructions

(1) Write down the names of the people you worked with.
(2) Write down any resources you used other than ones that most of your classmates would be familiar with, such as the textbook or Wikipedia.
(3) Write down the number of hours it took you to complete this assignment.
(4) Write your name, Math 4171, and the homework number.
(5) Hand in your homework in class.

Problems

(1) This question is about the *separation axioms*, some of which are listed below.
- In class, we defined a space to be T_0 (Kolmogorov) if, for any two distinct points x and y, there is either an open set that contains x but not y, *or* an open set that contains y but not x.
- A space is T_1 (accessible or Fréchet) if, for any two distinct points x and y, there is an open set that contains x but not y *and* an open set that contains y but not x.
- In class we defined a space to be T_2 (Hausdorff) if, for any two distinct points x and y, there are disjoint open neighborhoods U and V of x and y, respectively.
- A space is T_3 (regular Hausdorff) if it is T_0 and, for any closed set C and any $x \notin C$, there are disjoint open neighborhoods of x and C. That is, there are disjoint open sets U and V such that $x \in U$ and $C \subseteq V$.
- A space is T_4 (normal Hausdorff) if it is T_1 and, for any disjoint closed sets C and D, there are disjoint open neighborhoods of C and D. That is, there are disjoint open sets U and V such that $C \subseteq U$ and $D \subseteq V$.

(a) Find a space that is not T_0. Justify your claim. Hint: We've seen an example.
(b) Find a space that is T_0 but not T_1. Justify your claim. Hint: See page 76 of Munkres.
(c) Find a space that is T_1 but not T_2. Justify. Hint: You've seen an example.
(d) It is clear from the definitions that if a space is T_2, then it is T_1, and if a space is T_1, then it is T_0. Show that if a space is T_3, then it is T_2.
(e) One might wonder why the definition of T_3 assumes that the space is T_0. Find a space that is not T_0 that would otherwise satisfy the definition of T_3. Justify your claim. Hint: Chances are, your example from part 1a will work.
(f) Show that X is T_1 if and only if for any $x \in X$, the singleton set $\{x\}$ is closed. Conclude that T_4 implies T_3.
(g) One might wonder if the T_1 assumption in the definition of T_4 is necessary. Find a T_0 space that is not T_1 that would otherwise satisfy the definition of T_4. Justify your claim. Hint: Chances are, your example from part 1b will work.

(2) In this problem, we will show that $\mathbb{R}^n$ with the standard topology is T_4. (Note that the previous version of the problem had $\mathbb{R}$ instead of $\mathbb{R}^n$, but your proof should be pretty much the same.)

(a) Let C and D be disjoint closed subsets of $\mathbb{R}^n$. For any $c \in C$, show that there exists an ε_c such that the ball $B_{\varepsilon_c}(c)$ is disjoint from D. Similarly, for any $d \in D$, there is a ball $B_{\varepsilon_d}(d)$ is disjoint from C.

(b) If we naively took the unions of these open balls, the resulting open sets might not be disjoint. Also, we haven't really used the notion of distance, which will have to come into play, otherwise we might accidentally prove the false claim that every topological space is T_4. The key step is to halve the radii of the balls, and define

$$U = \bigcup_{c \in C} B_{\varepsilon_c/2}(c), \qquad V = \bigcup_{d \in D} B_{\varepsilon_d/2}(d).$$

Show that U and V are disjoint open neighborhoods of C and D.

(c) Check that $\mathbb{R}^n$ is T_1, and conclude that $\mathbb{R}^n$ is T_4.

(3) If $f\colon X \to Y$, then f is called open if $f(U)$ is open in Y for all U in X, and f is called closed if $f(C)$ is closed in Y for all C in Y. Construct a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ whose image is the half-open interval $(0, 1]$. Conclude that continuous functions might be neither open nor closed. You should justify that your function is continuous, but you may use basic facts from analysis, for example that the sum, product, composition, and so forth of continuous functions are continuous, and that standard functions are continuous.

(4) (a) Show that $f\colon X \to Y$ is continuous if and only if the preimage of any closed set is closed.

(b) Since the subspace topology is the smallest topology so that the inclusion map is continuous, and the quotient topology is the largest topology so that the quotient map is continuous, conclude that the definitions of the subspace of topology and the quotient topology in terms of closed sets are analogous to the definitions in terms of open sets.

(5) A common theme in geometry and topology is whether we can test a global property by testing it locally. For example, we can't test if a space is or is not $\mathbb{R}^2$ locally, because every little piece of $\mathbb{R}^2$ looks exactly like a little piece of the cylinder $S^1 \times \mathbb{R}$. Even if we look at every little piece, unless we know how to glue them together, we won't know whether we're looking at the plane or the cylinder.

In this question, we will show that we can test continuity locally. Let $f\colon X \to Y$. We call a collection of sets $\{U_\alpha\}$ an *open cover* of X if, as one might expect, the sets U_α are open, and they cover X, that is, $\bigcup_\alpha U_\alpha = X$. Open covers will be super important later. Think of the U_α as our "little pieces" of X.

Show that $f\colon X \to Y$ is continuous if and only if every restriction $f|_{U_\alpha}\colon U_\alpha \to Y$ is continuous.

(6) In the previous homework, you showed that a subspace of a T_2 (Hausdorff) space is T_2. Convince yourself that the proof for the corresponding claim for T_1 and T_0 is similar. In addition, a subspace of a T_3 space is T_3, but a subspace of a T_4 space *need not* be T_4, though we don't yet have the tools to produce a counterexample.

(a) Prove that a subspace of a T_3 space is T_3.

(b) Which part of this proof fails if you try to use it to prove that a subspace of a T_4 space is T_4?

(c) Prove that a *closed* subspace of a T_4 space is T_4.

(7) Do exercise 3.3 on page 28 of Munkres.

(8) Do exericse 3.4 on page 28 of Munkres. (We can think of this as the reverse of the quotient map construction. In this exercise, given the quotient map, we construct the equivalence relation.)

(9) (a) Generalizing the first part of the previous exercise, let $f\colon A \to B$ be an arbitrary function, and let $\sim_B$ be an equivalence relation on B. We *pullback* the equivalence relation on B to define a relation $\sim_A$ on A by $a \sim_A a'$ if and only if $f(a) \sim_B f(a')$. Show that $\sim_A$ defines an equivalence relation on A. Again, if you'd like, you can do

the first part of the previous exercise by citing this one appropriately, but then your points on that exercise rely on getting this one correct.

(b) We could try the other direction as well. Given an equivalence relation $\sim_A$ on A, define the *pushforward* relation $\sim_B$ on B by $f(a) \sim_B f(a')$ whenever $a \sim_A a'$. More precisely, $b \sim_B b'$ if and only if there exist $a, a' \in A$ such that $b = f(a)$, $b' = f(a')$, and $a \sim_A a'$. However, we run into problems.

   (i) Show that $\sim_B$ is reflexive if and only if f is surjective.

   (ii) Assuming f is surjective, must $\sim_B$ be an equivalence relation? If yes, prove it. If not, provide a counterexample, and then write down an additional assumption under which you can prove that $\sim_B$ would necessarily be an equivalence relation.

(10) Quotients and separation axioms do not like each other as much. Let $X = \mathbb{R}$.

(a) Consider the equivalence relation defined by $x \sim y$ if either x and y are both rational or x and y are both irrational. Compute $X/\sim$ and describe its quotient topology. Conclude that the quotient of a T_4 space need not be T_0.

(b) If we replace the rationals $\mathbb{Q}$ in the above example with the closed interval $[0, 1]$, describe the quotient topology $X/\sim$. Which T_i spaces is it in?

(11) Let X be a topological space with an equivalence relation $\sim$. Assume that every equivalence class is closed in X. This was the case in our example from class on $\mathbb{R}^2$, for example. Show that $X/\sim$ is T_1, (regardless of whether X satisfies any of the separation axioms). Conversely, show that if $X/\sim$ is T_1, then every equivalence class is closed in X.