TOPOLOGY HOMEWORK 4, DUE SEPTEMBER 28

Instructions

(1) Write down the names of the people you worked with.
(2) Write down any resources you used other than ones that most of your classmates would be familiar with, such as the textbook or Wikipedia.
(3) Write down the number of hours it took you to complete this assignment.
(4) Write your name, Math 4171, and the homework number.
(5) Hand in your homework in class.

Problems

(1) (a) Show that A is open in $\mathbb{R}^n$ if and only if $A \times \mathbb{R}^k$ is open in $\mathbb{R}^n \times \mathbb{R}^k$. Hint: To simplify notation, use the notation $(x, y) \in \mathbb{R}^n \times \mathbb{R}^k$ rather than $(x_1, \ldots, x_n, y_1, \ldots, y_k)$.
(b) Consider the equivalence relation on $\mathbb{R}^n \times \mathbb{R}^k$ where $(x, y) \sim (x, y')$ for all $x \in \mathbb{R}^n$ and $y, y' \in \mathbb{R}^k$. Show that $(\mathbb{R}^n \times \mathbb{R}^k)/\sim$ is homeomorphic to $\mathbb{R}^n$.

(2) (a) Let $A \subseteq \mathbb{R}^2$. Let the cone $C(A)$ be

$$C(A) = \{\lambda(x, y, 1) \mid (x, y) \in A, \lambda \in \mathbb{R} \setminus \{0\}\}.$$

Show that A is open in $\mathbb{R}^2$ if and only if $C(A)$ is open in $\mathbb{R}^3 \setminus \{0\}$. Hint: It turns out to be somewhate intricate to find an ε-ball around a point in $C(A)$. Instead, construct a function $f\colon \{(x, y, z) \mid z \neq 0\} \to \mathbb{R}^2$ that is continuous based on what you know from analysis, and such that, for all A, $C(A)$ is the preimage of A.
(b) Use the definition of the projective plane, $\mathbb{RP}^2 = (\mathbb{R}^3 \setminus \{0\})/\sim$, where $(x, y, z) \sim \lambda(x, y, z)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$. Again, we use the shorthand $[x : y : z]$ for $[(x, y, z)]$. Consider the map $f_z\colon \mathbb{R}^2 \to \mathbb{RP}^2$ defined by

$$f_z(x, y) = [x : y : 1].$$

Let $U_z = f_z(\mathbb{R}^2)$. Show that U_z is open in $\mathbb{RP}^2$.
(c) Show that f_z is a homeomorphism onto its image. That is, $f_z\colon \mathbb{R}^2 \to U_z$ is a homeomorphism, where U_z has the subspace topology. (In this case, the subspace topology is easy to describe because U_z is open.)
(d) Let $C_z = \mathbb{RP}^2 \setminus U_z$. Write down a natural bijection $\mathbb{RP}^1 \to C_z$, and show that it is a homeomorphism, where C_z has the subspace topology. Consequently, we can view $\mathbb{RP}^2 = U_z \cup C_z = \mathbb{R}^2 \cup \mathbb{RP}^1$, that is, the plane with an $\mathbb{RP}^1$ at infinity.
(e) Define $f_y\colon \mathbb{R}^2 \to \mathbb{RP}^2$ similarly, by $f_y(x, z) = [x : 1 : z]$, and likewise for U_y, f_x, and U_x. Show that $U_x \cup U_y \cup U_z$ is an open cover of $\mathbb{RP}^2$. What's missing from $U_x \cup U_z$?

(3) Do problem 3.12 on page 29. An *immediate predecessor* of an element b is an element a less than b such that there are no elements in between, that is, there does not exist a z such that $a < z < b$. Two ordered sets have the same *order type* if there is an order-preserving isomorphism between them. We expect them to have the same properties as far as order is concerned. Depending on how comfortable you feel with the previous statement, you may want to explicitly prove by contradiction that there is not an order isomorphism between any two of the orders in the problem.

(4) Do problem 3.13 on page 29.

(5) (a) Let X be a topological space. Show that closed sets are closed under taking limits. That is, let C be closed in X, and let $x_i \in C$ be a sequence that converges to $x \in X$. Show that $x \in C$.
(b) Now let $A \subseteq X$, and let $\overline{A}$ be its closure. Conclude that if $x_i \in A$ and the x_i converge to $x \in X$, then $x \in \overline{A}$.

(6) Do problem 17.6 on page 101 of Munkres.

(7) Do problem 17.8 on page 101 of Munkres. Include any counterexamples needed to justify that your answers are correct.

(8) Do problem 17.19 on page 102 of Munkres. Here, $\operatorname{Int} A$, the *interior* of A, is the corresponding notion to closure but with open sets. Instead of the intersection of closed sets containing A, the interior $\operatorname{Int} A$ is the *union* of all open sets *contained in* A.