TOPOLOGY HOMEWORK 5, DUE OCTOBER 3

Instructions

(1) Write down the names of the people you worked with.
(2) Write down any resources you used other than ones that most of your classmates would be familiar with, such as the textbook or Wikipedia.
(3) Write down the number of hours it took you to complete this assignment.
(4) Write your name, Math 4171, and the homework number.
(5) Hand in your homework in class.

Problems

(1) In class, we defined a total order $\leq$ by the partial order axioms of reflexivity, antisymmetry, and transitivity, along with the totality axiom that any two elements are comparable. We then declared $x < y$ to mean $x \leq y$ but $x \neq y$. Meanwhile, on page 24, Munkres defines an order relation $x < y$, and he restates the axioms again more sensibly on the top of page 25. Then, he defines $x \leq y$ as $x < y$ or $x = y$. Show that these two sets of axioms are equivalent. Make sure it's clear in your work which set of axioms you're assuming and which set you're proving, and make sure to do both directions.

(2) Do problem 17.20 on page 102 of Munkres, parts (a) through (c). I will assign parts (d) through (f) next time, so if you're on a roll, you can keep going on this problem, but use a separate sheet of paper. Remember that Munkres uses $x \times y$ to mean the ordered pair (x, y) in a way that's not ambiguous with interval notation. If you claim that any sets are open or closed, justify your claim, but remember our trick from class of constructing an obviously continuous map so that our set of interest is an inverse image of a set known to be open or closed. You may assume that points, closed intervals, and closed rays are known to be closed in the real line, and open intervals and rays are known to be open, though you should know how to prove these claims from the definition of the topology on $\mathbb{R}$. Likewise, you may boldly claim that sequences in Euclidean space converge as you expect, though you should again know how to prove these claims from the definitions.

(3) In section 22, Munkres talks about quotients in a different way than what we did in class. We'll show that these notions are equivalent so that you have a second way of thinking about quotients. Munkres defines a map $p\colon X \to Y$ between topological spaces to be a *quotient map* if it is surjective and we have that U is open in Y if and only if $p^{-1}(U)$ is open in X. (In particular, p must be continuous, but the condition is stronger than that.)

  (a) Let X be a topological space, let $\sim$ be an equivalence relation on X, and let $X/\!\sim$ be the quotient space as defined in class. Follow the definitions to show that $\pi\colon X \to X/\!\sim$ is a quotient map.

  (b) Now let X and Y be topological spaces, and let $p\colon X \to Y$ be a quotient map. Use the map p to define an equivalence relation $\sim$ on X as in Munkres exercise 3.4 on page 28, which you did a couple weeks back. Then, we can define a quotient space $X/\!\sim$ using the definition from class. Show that $X/\!\sim$ is homeomorphic to Y.
  We can conclude that if Y is just a set and $p\colon X \to Y$ is surjective, then there is a unique topology on Y that makes p a quotient map, which is the way in which Munkres defines the quotient topology.

(4) Let A be a subset of a topological space X. A common equivalence relation we can define in this situation is $x \sim y$ if either $x = y$ or both x and y in A. The intuition is that we are collapsing A to a point. We use the notation X/A to denote $X/\sim$. An example we did in class can be rephrased this way as $[0,1]/\{0,1\} \cong S^1$.

  (a) Show that, as *sets*, X/A is isomorphic to $(X \setminus A) \cup \{*\}$, where $*$ denotes a new point.

  (b) Describe the topology on X/A, viewing it as the same set as $(X \setminus A) \cup \{*\}$. Equivalently, consider the map $X \to (X \setminus A) \cup \{*\}$ and describe the quotient topology on $(X \setminus A) \cup \{*\}$ in view of Munkres's definition. That is, let $U \subseteq (X \setminus A) \cup \{*\}$. When is U open in the quotient topology? Let $V \subseteq X \setminus A$, and consider two cases, one where $U = V$, and one where $U = V \cup \{*\}$. In each case, determine when U is open in the quotient topology in terms of A, V and the topology on X.

(5) We explore some examples of X/A.

  (a) Show that $\mathbb{R}/[-1,1]$ is homeomorphic to $\mathbb{R}$.

  (b) Show that $\mathbb{R}/(-1,1)$ is not T_1 by demonstrating two points such that any open neighborhood of one of them contains the other.

  (c) What subset of $\mathbb{R}$ is $\mathbb{R}/(-\infty, 0]$ homeomorphic to? Prove your answer.

(6) (a) Show that if X is T_4 and A is closed, then X/A is T_4. Hint: Your proof should fail if $X = \mathbb{R}$ and $A = \mathbb{Q}$. Think about the two closed sets $\{\pi\}$ and $\{e\}$ in $\mathbb{R}/\mathbb{Q}$, for example.

  (b) What part of the proof fails if you replace T_4 by T_3?

(7) Going back to a general quotient space, let X be a connected topological space, and let $\sim$ be an equivalence relation. Show that $X/\sim$ is connected. If you need to remind yourself of the definition of a connected space, see page 148 of Munkres.