TOPOLOGY HOMEWORK 6, DUE OCTOBER 12

(1) Do problem 17.20 on page 102 of Munkres, parts (d) through (f). Again, make sure to justify that sets you claim to be open or closed are open or closed.

(2) Recall that we defined a space X to be T_6 (perfectly normal Hausdorff) if every closed set C is a zero set, that is, $C = f^{-1}(0)$ for some continuous function $f\colon X \to \mathbb{R}$. I neglected to mention that, as with T_4, we must additionally explicitly assume that the space is T_1, though this is not particularly relevant for this problem. Without the T_1 assumption, the space is just called perfectly normal.

A more standard definition of T_6 is that a space is T_6 if it is T_1 and for any two disjoint closed sets C and D, there exists a continuous function $h\colon X \to \mathbb{R}$ such that $h^{-1}(0) = C$ and $h^{-1}(1) = D$. This definition puts T_6 in line with the other separation axioms, and we can think of it as saying that disjoint closed sets can be separated by continuous functions. We will show that these two definitions are equivalent. One direction is easy, because, in particular, C is a zero set of g. We'll prove the other direction.

(a) Let C and D be disjoint closed subsets of X. By assumption, let $f, g\colon X \to \mathbb{R}$ be continuous such that $C = f^{-1}(0)$ and $D = g^{-1}(0)$. Explain why we can assume without loss of generality that f and g are nonnegative. Hint: absolute value.

(b) Let $h = \frac{f}{f+g}$. Show that h satisfies the requirements of the new definition of T_6 for the closed sets C and D, completing the proof of the equivalence of these definitions.

(c) What part of your proof fails if C and D are not disjoint?

(d) Conclude that the T_6 (perfectly normal Hausdorff) property implies the T_4 (normal Hausdorff) property.

(3) A property of a topological space is called *hereditary* if any subspace of a topological space with this property also has this property. In other words, if X has this property, then any subspace of X inherits this property from X. We showed earlier that T_0, T_1, T_2, and T_3 were hereditary, but we were unable to prove that T_4 was hereditary, and we'll eventually see a counterexample.

(a) Show that the T_6 property is hereditary.

(b) A space X is called hereditarily normal Hausdorff (T_5) if any subspace of X is T_4. Effectively, T_5 is the version of T_4 that is hereditary by fiat, and is also called completely normal Hausdorff. Conclude that, in fact, the T_6 property implies the T_5 property.

(4) We will investigate the one-dimensional modern Zariski topology. For technical reasons, we will use $\mathbb{C}$ instead of $\mathbb{R}$, but you can imagine that we're working with $\mathbb{R}$ and nothing will go wrong. (The full definition of the Zariski topology, which we won't cover, becomes wonky, and hence a promising field of research, if the field is not algebraically closed.) In the classical Zariski topology, a set is closed in $\mathbb{C}$ if it is the zero set $Z(\mathcal{F})$ of a set of polynomials $\mathcal{F}$. As discussed in class, in one dimension, we get the cofinite topology on $\mathbb{C}$. Also as discussed in class, such a zero set is an *irreducible variety* if it is nonempty and cannot be decomposed as the union of two proper closed subsets (not necessarily disjoint). The modern Zariski topology is defined on a larger set X, which has a point p for every irreducible variety. The topology is defined the same way, where $p \in Z(f)$ now means that f is identically zero on the variety corresponding to the point p.

(a) Show that, as a set, in the one-dimensional case, $X \cong \mathbb{C} \cup \{*\}$. That is, show that, in one dimension, there is a irreducible variety for every complex number a, and exactly one additional irreducible variety.
(b) Describe the topology on X in terms of this bijection to $\mathbb{C} \cup \{*\}$.
(c) Viewing $\mathbb{C}$ as a subspace of X, what is its subspace topology?
(d) Recall that the specialization partial order is defined by $p \leq q$ whenever $p \in \overline{\{q\}}$. What is the specialization order on X?

(5) Recall that a collection of open sets $\{U_\alpha \mid \alpha \in I\}$, where I is some index set, is called an open cover of X if the union of these open sets is all of X. Sometimes, such a cover is redundant, and so an open subcover is a subcollection $\{U_\alpha \mid \alpha \in I'\}$, where $I' \subseteq I$, that is still an open cover of X. Note that we're taking a subset of the collection, not subsets of the open sets themselves. A space X is called *compact* if every open cover of X has a *finite* subcover. This definition will be crucial later on.

A less crucial but nonetheless important definition is that of a *Noetherian space*. In class, we defined a space to be Noetherian if, for any descending chain of closed sets $C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots$, all but finitely many of the closed sets are equal to each other, or, in other words, the sequence eventually becomes constant. The classical Zariski topology was an example.

Show that any Noetherian space is compact. Warning: You should not assume that the open cover $\{U_\alpha\}$ is countable. Instead, pick a set U_{α_0} in the collection. Then, decide how you're going to pick the set U_{α_1} out of the collection. Then, based on that, decide how you're going to pick the set U_{α_2} out of the collection. For simplicity, it's okay to denote these by U_0, U_1, and U_2. As in induction, once you've picked sets $U_0, \ldots, U_n$ out of the collection, write down how (and if) you're going to pick U_{n+1}.

(6) In class, we discussed that, intuitively, $[0,1]/\{0,1\} \cong S^1$, where the easiest way to think of the circle is as the unit complex numbers as a subspace of $\mathbb{C}$. We'll show that this is in fact the case.
(a) Prove that the open interval $(0,1)$ is homeomorphic to $S^1 \setminus \{1\}$. Hint: For the less straightforward direction, start with the fact that the complex logarithm is continuous on the open set $\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\} \to \{z \in \mathbb{C} \mid \operatorname{Im} z \in (-\pi, \pi)\}$. (To prove that, you'd make use of the inverse function theorem from analysis on e^z.)
(b) Prove that $[0,1]/\{0,1\} \cong S^1$.
(c) Let n be a positive integer, and define an equivalence relation on S^1 by $\theta \sim \theta'$ if $\theta - \theta'$ is a multiple of $2\pi/n$, where θ is an angle parametrizing S^1. Show that $S^1/{\sim} \cong S^1$. In other words, we just get a circle that's "n times smaller."
Suggested strategy: Use complex numbers to write down a quotient map $p\colon S^1 \to S^1$. For the tricky part of showing that $p^{-1}(U)$ is open implies that U is open, consider two cases. In one case, $U = S^1$, and in the other $U \neq S^1$. The first case is easy, and explain why, in the second case, you can assume without loss of generality that $1 \notin U$. At this point, it's good to get an intuitive sense of what's going on. Get some scratch paper, draw some circles, pick a set U, and draw its inverse image, say, for $n = 5$. At this point, you should have an intuitive argument for why $p^{-1}(U)$ is open implies that U is open. To formalize this argument, use part 6a, which gives you a way of converting open sets in $S^1 \setminus \{1\}$ to open sets in $(0,1)$ and back. It's easier to prove things about sets in $(0,1)$.