TOPOLOGY HOMEWORK 7, DUE OCTOBER 19

(1) Do problem 4.4 on page 34 in Munkres.

(2) Do problem 4.9 on page 35 in Munkres.

Warning: You want to prove part (d) from part (c), which you prove from part (b), which you prove from part (a). Don't implicitly assume the later parts when doing earlier parts. Instead, rely on the basic properties of $\mathbb{N}$ of well-ordering and the Archimedean property, the definition that $\mathbb{Z}$ contains zero, the elements of $\mathbb{N}$, and their negatives, along with things claimed in the text such as that $\mathbb{Z}$ is closed under addition, subtraction, and multiplication, and that if n is an integer there is no integer strictly between n and $n+1$.

Suggested strategy for part (a): Let S be the set. The given upper bound might not be an integer, but use the Archimedean property to conclude that there is an integer n strictly larger than anything in S. Then consider $n - S = \{n - s \mid s \in S\}$. Conclude that it is a nonempty subset of $\mathbb{N}$, and then use the well-ordering property.

For part (b), don't forget to use the Archimedean property to check that the set you define is nonempty, and don't forget to show that there is *exactly* one n. For part (c), don't forget the case where x is an integer.

(3) Do problem 5.4 on page 39 of Munkres. Munkres uses the notation X^ω to refer to the set of sequences in X, which we called $X^{\mathbb{N}}$ in class.

(4) Do problem 5.5 on page 39 of Munkres.

(5) Do problem 10.1 on page 66 of Munkres.

(6) Do problem 10.2 on page 67 of Munkres.

(7) In this problem, we will investigate arithmetic operations on well-ordered sets, that is, sets satisfying the well-ordering property on page 32, beginning with addition. This problem has some overlap with Munkres 10.8.

   (a) Let A and B be sets with a total order. Let $A \sqcup B$ be their disjoint union. (There are formal ways to define this with tuples, but basically we declare elements of A to be different from elements of B, so $\mathbb{N} \sqcup \mathbb{N}$ contains two distinct copies of every natural number, for example.) We let $A + B$ denote $A \sqcup B$ with the order where, intuitively, we put A to the left of B. Formally, for $a_1, a_2 \in A$ we have $a_1 \leq_{A+B} a_2$ whenever $a_1 \leq_A a_2$. Likewise, $b_1, b_2 \in B$ we have $b_1 \leq_{A+B} b_2$ whenever $b_1 \leq_B b_2$. Finally, for all $a \in A$ and $b \in B$, declare $a \leq_{A+B} b$.
   Show that if A and B are well-ordered then so is $A + B$.

   (b) Let $\{*\}$ be the set containing one element. Find an order-preserving isomorphism between $\{*\} + \mathbb{N}$ and $\mathbb{N}$.

   (c) Show that there is no order-preserving isomorphism between $\mathbb{N} + \{*\}$ and $\mathbb{N}$. Thus, we have an exception to my claim from class that anything called addition should be commutative. Suggestion: For this question and later on, to show that two sets have different order type, adapt the techniques of problem 3.12. Since these are all well-ordered, they will have least elements and immediate successors based on your earlier work, but greatest elements and immediate predecessors are still good tests.

(8) We'll now move on to multiplication.

   (a) We already discussed the lexicographic order on $A \times B$. Show that if A and B are well-ordered, then so is $A \times B$. Try it out on your own, but if needed see Munkres prove this fact in the special case $\mathbb{N} \times \mathbb{N}$ on page 63.

(b) Find an order-preserving isomorphism between $\mathbb{N} \times \{0,1\}$ and $\mathbb{N}$.
(c) Show that there is no order-preserving isomorphism between $\{0,1\} \times \mathbb{N}$ and $\mathbb{N}$. This is effectively problem 10.3 on page 67 of Munkres.

(9) Finally, we'll talk about exponentiation, which doesn't work out quite as nicely. It works better if we change our definition of exponentiation for well-ordered sets, but we're not going to do that.
(a) If A has a total order and I is well-ordered, show that you can define a total "lexicographic" order on A^I, the set of functions from I to A by formalizing the intuition of looking at the least index where they differ. Check that the total order axioms are satisfied. Warnings: Make sure your proof fails when $I = \mathbb{R}$, since it's not clear how to compare $\sin x$ and 0. Also, make sure your proof doesn't implicitly assume that $I = \mathbb{N}$. Well-ordered sets can get pretty weird, so just rely on the well-ordering property.
(b) Even if A is well-ordered, A^I is generally not. Consider $\{0,1\}^{\mathbb{N}}$, that is, sequences of zeroes and ones. Let s_n be the sequence with zeroes everywhere except for a one in the nth position. With the order you defined above, show that $\{s_n \mid n \in \mathbb{N}\}$ has no least element.