TOPOLOGY HOMEWORK 8, DUE OCTOBER 26

(1) We discussed two definitions of when a field F is an ordered field. In Munkres's definition, an ordered field is a field with a total order relation $<$ satisfying the compatibility properties (6) on page 31. Another definition is that an ordered field has a subset P of positive numbers that is closed under addition and multiplication and satisfies the trichotomy axiom that, for any $x \in F$, exactly one of the three following statements is true: $x \in P$, $-x \in P$, and $x = 0$.

In the below exercises, you may use basic properties of fields, see for example exercise 4.1 on page 34, but, as you'd expect, you can't use basic properties of ordered fields, see for example exercise 4.2.

(a) Assume that a field F has a total order satisfying Munkres's defintion of an ordered field. Let $P = \{x \in F \mid 0 < x\}$. Show that P satisfies the above axioms of positive numbers above.

(b) Now assume that F has a subset of positive numbers P satisfying the axioms above. Define a relation $<$ by $x < y$ whenever $y - x \in P$. Show that $<$ is a total order and that it satisfies Munkres's compatibility axioms.

(2) Let A be a set with a total order. Recall that a set L is a lower set if for all $a \in L$, if $b \leq a$, then $b \in L$ also. Let $\mathcal{L}$ be the set of all lower sets in A that are proper subsets of A.

(a) Show that if L and M are two lower sets, then either $L \subseteq M$ or $M \subseteq L$. Consequently, the inclusion order $\subseteq$, normally only a partial order, is a total order on $\mathcal{L}$. Warning: Don't implicitly assume that A has the least upper bound property.

(b) If A is well-ordered, show that there is an order isomorphism between $\mathcal{L}$ and A. Warning: Double check that your argument works for $A = \{0\}$ and $A = \mathbb{N}$.

(c) If $A = \mathbb{R}$, show that there is an order isomorphism between $\mathcal{L}$ and $\{*\} + \mathbb{R} \times \{0, 1\}$. That is, give $\mathbb{R} \times \{0, 1\}$ the lexicographic order, and then add an additional special element $\{*\}$ that is smaller than all other elements.

(3) On page 76, Munkres draws nine different topologies on the set with three elements $X = \{a, b, c\}$. Munkres draws the open sets, but *for this problem treat the drawn sets as closed.* (Since the set is finite, there is no distinction between finite and arbitrary unions, so the topological axioms for open sets are the same as those for closed sets.) Recall the specialization order $x \leq y$ whenever $x \in \overline{\{y\}}$. Compute the specialization preorder for the nine topologies drawn. For consistency, please do them in the following order:

I	IV	VII
II	V	VIII
III	VI	IX

(4) Do exercise 13.4(c) in Munkres. Recall that Munkres uses the notation $\mathcal{T}$ to refer to the set of open subsets of a topology, which he calls "the topology."

(5) Do exercise 13.5 in Munkres.

(6) Do exercise 13.6 in Munkres. The lower limit topology $\mathbb{R}_l$ and the topology $\mathbb{R}_K$ are defined on pages 81–82. Warning: It's not enough to say that the bases are not comparable. In $\mathbb{R}^2$, the the disk basis and the rectangle basis are disjoint, but they generate the same topology.

(7) Do exercise 13.7 in Munkres.

(8) Do exericse 13.8 in Munkres. For part (b), don't forget to check that the given collection is a basis.

(9) In class, we discussed the *universal property of a product*: The product $X \times Y$ has maps $\pi_X \colon X \times Y \to X$ and $\pi_Y \colon X \times Y \to Y$, and for any pair of maps $f \colon Z \to X$ and $g \colon Z \to Y$, there exists a unique map $f \times g \colon Z \to X \times Y$, such that $\pi_X \circ (f \times g) = f$, and $\pi_Y \circ (f \times g) = g$.

Working directly from the universal property, in particular, without writing ordered pairs, show that, given maps $f \colon Z \to X$ and $g \colon W \to Y$, there is a unique map $f \hat{\times} g \colon Z \times W \to X \times Y$ such that the following two diagrams commute.

$$\begin{array}{ccc} Z \times W & \xrightarrow{f \hat{\times} g} & X \times Y \\ \downarrow{\scriptstyle \pi_Z} & & \downarrow{\scriptstyle \pi_X} \\ Z & \xrightarrow{\quad f \quad} & X \end{array} \qquad \qquad \begin{array}{ccc} Z \times W & \xrightarrow{f \hat{\times} g} & X \times Y \\ \downarrow{\scriptstyle \pi_W} & & \downarrow{\scriptstyle \pi_Y} \\ W & \xrightarrow{\quad g \quad} & Y \end{array}$$

The advantage of using the universal property is that you never have to talk about open sets or continuity.