Topology I Midterm 2

November 9, 2016

Name: ______________________________

- There are four questions, each worth twenty-five points, so you should pace yourself at around 12–15 minutes per question, since they vary in difficulty and you'll want to check your work.
- Use the back of the previous page for scratchwork. By default, I won't grade the scratchwork, so you can write wrong things there without penalty.
- If you run out of space on the printed page and need more space, then use the back of the previous page, but make sure to:
  - Make a note on the printed page that your work continues on the back of the previous page.
  - On the back of the previous page, put a box around the work that you want graded.
- Give and use definitions from the book or from class.
- You may use any results you remember from the book or from class as long as they are more basic than the result you're asked to prove.

1. (a) (6 points) Let X be a set. Define a basis for a topology on X.

   (b) (6 points) Define the topology generated by a basis.

   (c) (6 points) Let X be the set $\mathbb{R}$, and let $\mathcal{B}$ be the set of all closed intervals $[a, b]$ with $a < b$. Decide with proof whether $\mathcal{B}$ is a basis.

   (d) (7 points) Determine, with proof, the smallest topology such that every set in $\mathcal{B}$ is open. It's one of the topologies we've seen in class.

2. (a) (6 points) Let X_α be a collection of topological spaces for $\alpha \in J$. Define the product topology on $\prod_\alpha X_\alpha$. Don't use the universal property definition.

   (b) (6 points) Let $f_\alpha \colon Z \to X_\alpha$ be a collection of continuous maps from a topological space Z. Let $f \colon Z \to \prod X_\alpha$ be defined by $f(z)_\alpha = f_\alpha(z)$, that is, the α-th coordinate of $f(z)$ is $f_\alpha(z)$. Show that f is continuous.

   (c) (7 points) For each $\beta \in J$, let $\pi_\beta \colon \prod_\alpha X_\alpha \to X_\beta$ be the projection map. Let U be a subset of $\prod_\alpha X_\alpha$. Consider the following two statements.

   - If U is open in the product topology on $\prod_\alpha X_\alpha$, then $\pi_\beta(U)$ is open in X_β.
   - If $\pi_\beta(U)$ is open in X_β for all β, then U is open in the product topology on $\prod_\alpha X_\alpha$.

   Prove the statement that is true.

   (d) (6 points) For the statement that is not true, provide a counterexample with justification. To clarify, you can pick your index set J, your spaces X_α, and your subset U. Then verify that for those choices, your chosen statement is false.

3. (a) (6 points) Let X be a set. Define a metric on X.

   (b) (6 points) Let d be a metric on X. Define the metric topology induced by d on X.

   (c) (6 points) Let X be a set, and let d_1 and d_2 be two metrics on X. Let $d = d_1 + d_2$. Show that d is also a metric on X.

   (d) (7 points) Let $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}$ be the topologies induced by d_1, d_2, and d, respectively, on X. Consider the following two statements.

   - $\mathcal{T}$ is the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$.
   - $\mathcal{T}$ is the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

   Prove the statement that is true.

4. Let $X = [0,1]^{\mathbb{N}}$, that is, sequences with values in $[0,1]$. Let $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$. Let

$$\rho(\mathbf{x}, \mathbf{y}) = \sup_{i \in \mathbb{N}} |x_i - y_i| .$$

(a) (8 points) Verify that ρ well-defined and a metric on X.

(b) (9 points) Let $[0,1]^\infty$ be the subset of X containing all sequences $\mathbf{x}$ such that $x_i = 0$ for all but finitely many i. Let

$$\mathbf{y} = \left(1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \dots\right) \in X.$$

Find, with proof, a sequence $\mathbf{x}_n$ that converges to $\mathbf{y}$ with respect to the metric topology induced by ρ, such that $\mathbf{x}_n \in [0,1]^\infty$ for all n.

(c) (8 points) Let

$$\mathbf{z} = (1, 0, 1, 0, 1, 0, \dots) \in X.$$

Show that there does not exist a sequence $\mathbf{x}_n$ that converges to $\mathbf{z}$ with respect to the metric topology induced by ρ, such that $\mathbf{x}_n \in [0,1]^\infty$ for all n.

Question	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	