$C^*$-algebras of amenable non-discrete groups and their AF-embeddability, or not

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Plan

1. Group $C^*$-algebras of locally compact groups
2. Approximately finite-dimensional $C^*$-algebras (AF-algebras)
3. AF-embeddability of $C^*(G)$ for some amenable Lie groups $G$
4. AF-embeddability fails badly for other amenable Lie groups

The group $C^*$-algebra of a locally compact group $G$

- $C_c(G) := \{ f : G \to \mathbb{C} \mid f \text{ is continuous and its support is compact} \}$
- There is a left-invariant Haar measure $\lambda$ on $G$:
  $$\int_G \varphi(ax) d\lambda(x) = \int_G \varphi(x) d\lambda(x) \text{ for } a \in G \text{ and } \varphi \in C_c(G)$$

Example: The Lebesgue measure $\lambda$ on the additive group $G = (\mathbb{R}^n, +)$.

- the regular representation $\lambda : C_c(G) \to B(L^2(G))$, $\lambda(f)\varphi = f \ast \varphi$,
where $(f \ast \varphi)(x) = \int_G f(a)\varphi(a^{-1}x) d\lambda(a)$

- $C^*(G) := \overline{\lambda(C_c(G))} \subseteq B(L^2(G))$ (the reduced $C^*$-algebra of $G$, actually)

- $G$ is called amenable if it has a left-invariant positive functional on the bounded continuous functions $C_b(G)$ that extends a Haar integral

$\mathcal{C}_c(G) \subsetneq \mathcal{C}_b(G) \subsetneq L^1(G)$ if $G$ is not compact
Motivation

For which locally compact groups $G$ is $C^*(G)$ a quasidiagonal $C^*$-algebra?

- $\pi: A \to B(\mathcal{H})$ is a diagonal $*$-representation if
  \[
  \pi(\cdot) = \begin{pmatrix}
  \pi_1(\cdot) & 0 & 0 \\
  0 & \pi_2(\cdot) & \ddots \\
  0 & \ddots & \ddots
  \end{pmatrix}
  \]
  with finite-dim. blocks

that is, $\pi(a)E_n = E_n\pi(a)$ for all $a \in A$,
where $E_1 + E_2 + \cdots = 1$ with $E_n = E_n^* = E_n^2 \in B(\mathcal{H})$, $\text{dim } E_n\mathcal{H} < \infty$.

- $\pi$ is a quasidiagonal $*$-representation if for $P_n = E_1 + \cdots + E_n$,
  \[
  (\forall a \in A) \lim_{n \to \infty} \|\pi(a)P_n - P_n\pi(a)\| = 0
  \]

- $A$ is quasidiagonal if it is separable and has a faithful quasidiagonal $*$-representation
Motivation (2)

Open problem (once again):
For which locally compact groups $G$ is $C^*(G)$ a quasidiagonal $C^*$-algebra?

Partial solution for countable discrete groups:

Step 1 (J. Rosenberg, ibid.): $G$ amenable $\iff C^*(G)$ quasidiagonal

\[ G \text{ amenable} \implies C^*(G) \text{ AF-embeddable} \quad (*) \]

Step 2b (Ch. Schafhauser, Ann. of Math., 2020):

\[ G \text{ amenable} \implies C^*(G) \hookrightarrow \bigotimes_{n \geq 1} M_n(\mathbb{C}) \text{ (trace-preserving embedding)} \]

Recall: AF-embeddable $\implies$ quasidiagonal $\implies$ stably finite

- The implication $(*)$ fails in general for non-discrete groups.
- Yet, it does hold true for many interesting examples of Lie groups.
Examples of group $C^*$-algebras

**Example 1:** $G$ compact group, e.g., $G = U(n) := \{ u \in M_n(\mathbb{C}) \mid u^* u = 1 \}$

Peter-Weyl theorem $\sim \quad C^*(G) \xrightarrow{\sim} \bigoplus_{\lambda \in \hat{G}} M_{d_{\lambda}}(\mathbb{C})$ (complete $c_0$-direct sum)

**Example 2:** $G = (\mathcal{V}, +)$ abelian group. The Fourier transform $
\mathcal{F} : \mathcal{S}(\mathcal{V}) \to \mathcal{S}(\mathcal{V}^*), \quad (\mathcal{F}(\varphi))(\xi) = \int_{\mathcal{V}} e^{-i\langle \xi, x \rangle} \varphi(x) dx$

satisfies $\mathcal{F}(\varphi_1 \ast \varphi_2) = \mathcal{F}(\varphi_1)\mathcal{F}(\varphi_2)$ and extends to a $*$-isomorphism $\quad C^*(\mathcal{V}, +) \xrightarrow{\sim} C_0(\mathcal{V}^*)$

We study group $C^*$-algebras of more general Lie groups. For instance, groups $G = (\mathcal{V}, \cdot)$ whose group operation $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is smooth and whose unit element is the zero $0 \in \mathcal{V}$ of the real vector space $\mathcal{V}$. 
Approximately finite-dimensional $C^*$-algebras

- Finite-dimensional $C^*$-algebra: $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$

- Approximately finite-dimensional $C^*$-algebra (AF-algebra): $A_1 \subseteq A_2 \subseteq \cdots \subseteq A$ with $\bigcup_{n \geq 1} A_n = A$ and $\dim A_n < \infty$ for $n = 1, 2, \ldots$

**Notation:** $A = \lim_{\longrightarrow} A_n$

**Fact:** closed $*$-subalgebra of an AF-algebra $\implies$ quasidiagonal $C^*$-algebra $\implies$ stably finite $C^*$-algebra:

$$v \in M_n(\tilde{A}) \quad \& \quad v^* v = 1 \implies vv^* = 1 \text{ for } n = 1, 2, \ldots$$

where $\tilde{A} := \begin{cases} \mathbb{C}1 + A & \text{if } 1 \not\in A, \\ A & \text{if } 1 \in A. \end{cases}$ (Every isometry in $M_n(\tilde{A})$ is unitary.)
Examples of AF-algebras

(a) \( \mathcal{K}(\mathcal{H}) = \lim_{\to} M_n(\mathbb{C}) \) via \( M_n(\mathbb{C}) \hookrightarrow M_{n+1}(\mathbb{C}), \ a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \)

(b) Cantor’s set \( K = \bigcap_{n=1}^{\infty} K_n, \ K_n = I_{n,1} \sqcup \cdots \sqcup I_{n,2^{n-1}} \subseteq [0, 1] = K_1 \)
\( \hookrightarrow \) the algebra of locally constant functions on \( K_n \) is \( C_{\text{loc}}(K_n) \sim \mathbb{C}^{2^n} \text{.} \)
\( \hookrightarrow \) \( C(K) = \lim_{\to} C_{\text{loc}}(K_n) \) where \( C_{\text{loc}}(K_n) \hookrightarrow C_{\text{loc}}(K_{n+1}), \ f \mapsto f|_{K_{n+1}} \)
\( \hookrightarrow \) Every abelian separable \( C^* \)-algebra is AF-embeddable
— in particular \( C^*(\mathcal{V}, +) \) is, for any finite-dimensional vector space \( \mathcal{V} \).

(b) \( G \) compact (second countable) group \( \iff \) \( C^*(G) \) is an AF-algebra
\( \hookrightarrow \) \( C^*(G) = \lim_{\to} M_{d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_n}(\mathbb{C}) \)
Primitive ideal space of a $C^*$-algebra $A$

- $\hat{A}$ the set of equivalence classes of the irreducible $*$-representations
- $\text{Id}(A)$ the closed two-sided ideals of $A$
- **topology on $\text{Id}(A)$**: a sub-base of closed sets consists of $\emptyset$ and the order intervals $\downarrow J := \{ I \in \text{Id}(A) \mid I \subseteq J \}$ for arbitrary $J \in \text{Id}(A)$
- $\text{Prim}(A) := \{ \text{Ker } \pi \mid [\pi] \in \hat{A} \} \subseteq \text{Id}(A)$ the primitive ideal space of $A$

\[ \geq \]
$\text{Prim}(A)$ is a $T_1$ topological space (i.e., its finite subsets are closed)

\[ \iff \]
every primitive ideal of $A$ is a maximal ideal

- $\text{Prim}(G) := \text{Prim}(C^*(G))$ if $G$ is a locally compact group
AF-embeddability for $C^*$-algebras of Lie groups

**Theorem**

If $G$ is a simply connected Lie group and $\text{Prim}(G)$ is $T_1$, then $C^*(G)$ is AF-embeddable.

**Main step of the proof (for the solvable radical $R \subseteq G$)**

If $R = (\mathcal{V}, \cdot)$ is a solvable Lie group and $\text{Prim}(R)$ is $T_1$, then $\text{Prim}(R)$ has no nonempty open compact subsets.

If $R = (\mathcal{V}, +)$ then $C^*(R) \simeq C_0(\mathcal{V}^*)$ and $\text{Prim}(R) \simeq \mathcal{V}^*$ is Hausdorff connected. If $R$ is noncommutative, then connectedness of $\text{Prim}(R)$ does not suffice.

For, $\text{Prim}(R)$ is not Hausdorff, so its compact subsets may not be closed.

**Recall:**

- $R$ group $\leadsto R^{(1)} :=$ the subgroup generated by $\{ghg^{-1}h^{-1} \mid g, h \in R\}$, furthermore $R^{(n+1)} := (R^{(n)})^{(1)}$ for $n = 1, 2, \ldots$

- $R$ is solvable $\iff R^{(n)} = \{1\}$ for some $n \geq 1 \implies R$ is amenable
Strategy of the proof

**Step 1**: Find a continuous open mapping $\text{Prim}(R) \to T$ with $T = [0, \infty)$ or $T = \mathbb{R}$. The construction is based on the following:

If $R$ is a simply connected solvable Lie group with its corresponding abelian group $R^{ab} := R/R^{(1)}$, then the dual group $\hat{R}^{ab}$ acts on $C^*(R)$ and there is a homeomorphism of quasi-orbit spaces $(\text{Prim}(R)/\hat{R}^{ab}) \sim \cong (R^{(1)}/R)\sim$.

**Step 1$rac{1}{2}$**: Any *nonempty open compact* subset of $\text{Prim}(R)$ would then be mapped onto a subset of $T$ with the same properties. But $T$ has no such subsets since $T$ is a connected noncompact Hausdorff space.

$\leadsto$ No nonempty open compact subsets of $\text{Prim}(R)$.

**Step 2**: If $A$ is nuclear separable and $\text{Prim}(A)$ has no nonempty open compact subset, then $A$ is AF-embeddable (J. Gabe, GAFA, 2020).

$\leadsto$ $C^*(R)$ is AF-embeddable if $R$ is a solvable Lie group

**Step 3**: $\text{Prim}(G)$ is $T_1 \implies G = R \rtimes K$ with $R$ as above and compact $K$
Examples

Example 1: If the Lie group $G$ is either compact or nilpotent (i.e., $G = (\mathcal{V}, \cdot)$ and $\cdot : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is a polynomial mapping), then $\text{Prim}(G)$ is $T_1$.

Example 2: Let $S \in M_d(\mathbb{R})$ and $G_S := \mathbb{R}^d \rtimes \mathbb{R}$, with its group operation $(x_1, t_1) \cdot (x_2, t_2) = (x_1 + e^{t_1 S} x_2, t_1 + t_2)$.

Then $G_S$ is a solvable Lie group and the following are equivalent:

1. Either $\Re z > 0$ for every $z \in \text{spec}(S)$ or $\Re z < 0$ for every $z \in \text{spec}(S)$.
2. The $C^*$-algebra $C^*(G_S)$ is not AF-embeddable/stably finite.
3. There exists a nonempty open compact subset of $\text{Prim}(G_S)$.

Moreover, $\text{Prim}(G_S)$ is $T_1$ if and only if $\text{spec}(S) \subseteq i\mathbb{R}$. 
A classical example: Mautner’s group

Let $\theta \in \mathbb{R}$, $S_{\theta} = \begin{pmatrix} 2\pi i & 0 \\ 0 & 2\pi i \theta \end{pmatrix} \in M_2(\mathbb{C}) \cong M_4(\mathbb{R})$ and the 5-dimensional solvable Lie group $G_{\theta} := G_{S_{\theta}} = \mathbb{C}^2 \rtimes \mathbb{R}$, with its group operation $(x_1, t_1) \cdot (x_2, t_2) = (x_1 + e^{t_1 S_{\theta}} x_2, t_1 + t_2)$, satisfies:

- $G_{\theta}$ is amenable and its center is trivial
- $\text{Prim}(G_{\theta})$ is $T_1$, no nonempty open compact subset
- $C^*(G_{\theta})$ is AF-embeddable
- If $\theta \in \mathbb{R} \setminus \mathbb{Q}$, then $C^*(G_{\theta})$ is antiliminary (i.e., it has no type I ideals).
Another classical example: Dixmier’s group

We consider the 3-dimensional Heisenberg group

$$H_3 = \left\{ T = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

and its representation

$$\pi : H_3 \to \text{End}_\mathbb{C}(\mathbb{C}^2), \quad \pi \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{ia} & 0 \\ 0 & e^{ib} \end{pmatrix}$$

The 7-dimensional solvable Lie group $G := \mathbb{C}^2 \ltimes H_3$ with its group operation $(x_1, T_1) \cdot (x_2, T_2) = (x_1 + \pi(T_1)x_2, T_1T_2)$ satisfies:

- $G$ is amenable and its center is trivial
- $\text{Prim}(G)$ is $T_1$, no nonempty open compact subsets
- $C^*(G)$ is AF-embeddable
- $C^*(G)$ is antiliminary